

# FLAG ALGEBRAS

Ping Hu

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Seminal paper:

Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.

David P. Robbins Prize of AMS for Razborov in  
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### EXAMPLE (GOODMAN, RAZBOROV)

If the density of edges is at least  $\rho > 0$ , what is the minimum density of triangles?

- Designed to attack extremal problems.
- Works well if constraints as well as desired value can be computed by checking small subgraphs (or average over small subgraphs).
- The results are for the limit as graphs get very large.

## FLAG ALGEBRAS DEFINITIONS

Let  $G$  be a 2-edge-colored complete graph on  $n$  vertices.



The probability that three random vertices in  $G$  span a red triangle, i.e.  $\# \triangle / \binom{n}{3}$ .

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The probability that a random vertex other than  $v$  is connected to  $v \in V(G)$  by a red edge, i.e., the red degree of  $v$  divided by  $n - 1$ .

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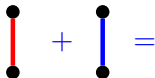
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$$\text{red edge} + \text{blue edge} = 1$$



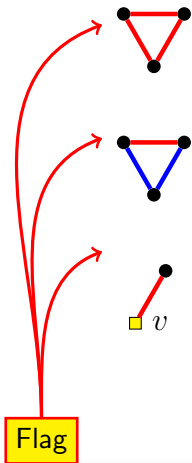
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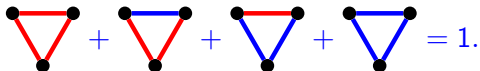
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*Type* is a flag induced by labeled vertices

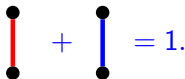
## FLAG ALGEBRAS IDENTITIES

Let  $G$  be a 2-edge-colored complete graph on  $n$  vertices. Then



$$\text{Red Triangle} + \text{Mixed Triangle} + \text{Mixed Triangle} + \text{Blue Triangle} = 1.$$

Same as



$$\text{Red Edge} + \text{Blue Edge} = 1.$$

## FLAG ALGEBRAS IDENTITIES

Let  $G$  be a 2-edge-colored complete graph on  $n$  vertices. Then by the law of total probability

$$\text{Diagram of a red edge} = \frac{3}{3} \text{Diagram of a red triangle} + \frac{2}{3} \text{Diagram of a triangle with 2 blue edges} + \frac{1}{3} \text{Diagram of a triangle with 1 blue edge} + \frac{0}{3} \text{Diagram of a triangle with 0 blue edges}.$$

Expanded version:

$$P\left(\text{Diagram of a red edge}\right) = P\left(\text{Diagram of a red edge} \mid \text{Diagram of a red triangle}\right) \cdot P\left(\text{Diagram of a red triangle}\right) + P\left(\text{Diagram of a red edge} \mid \text{Diagram of a triangle with 2 blue edges}\right) \cdot P\left(\text{Diagram of a triangle with 2 blue edges}\right) + \dots$$

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Let  $G$  be a 2-edge-colored complete graph on  $n$  vertices. Then


$$\begin{array}{c} \bullet \\ \textcolor{red}{\diagdown} \\ \square v \end{array} \times \begin{array}{c} \bullet \\ \textcolor{red}{\diagdown} \\ \square v \end{array} = \begin{array}{c} \bullet \quad \textcolor{gray}{?} \quad \bullet \\ \textcolor{red}{\diagdown} \quad \textcolor{red}{\diagup} \\ \square v \end{array} + o(1) = \begin{array}{c} \bullet \textcolor{red}{\text{---}} \bullet \\ \textcolor{red}{\diagdown} \quad \textcolor{red}{\diagup} \\ \square v \end{array} + \begin{array}{c} \bullet \textcolor{blue}{\text{---}} \bullet \\ \textcolor{red}{\diagdown} \quad \textcolor{red}{\diagup} \\ \square v \end{array} + o(1)$$

$o(1)$  as  $|V(G)| \rightarrow \infty$  (will be omitted on next slides)

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: The probability of choosing two different vertices ...


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
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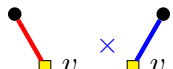
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 : The probability of choosing two different vertices ...

 : The probability that choosing two vertices  $u_1, u_2$  other than  $v$  gives red  $vu_1$  and blue  $vu_2$ .

$o(1)$  as  $|V(G)| \rightarrow \infty$  (will be omitted on next slides)

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Let  $G$  be a 2-edge-colored complete graph on  $n$  vertices. Then

$$\frac{1}{3} \text{ (triangle with 2 blue edges) } = \frac{1}{n} \sum_{v \in V(G)} \text{ (triangle with 2 blue edges and vertex } v \text{ highlighted) }$$



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$$\text{ (triangle with 2 red edges) } \binom{n}{3} = \sum_{v \in V(G)} \text{ (triangle with 2 red edges, vertex } v \text{ highlighted) } \binom{n-1}{2}$$

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$$\frac{1}{3} \begin{array}{c} \bullet \text{---} \text{blue} \text{---} \bullet \\ \diagdown \text{red} \diagup \\ \bullet \end{array} = \frac{1}{n} \sum_{v \in V(G)} \begin{array}{c} \bullet \text{---} \text{blue} \text{---} \bullet \\ \diagdown \text{red} \diagup \\ \text{yellow square } v \end{array}$$

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# IDENTITIES SUMMARY

Let  $G$  be a 2-edge-colored complete graph on  $n$  vertices. Then

$$1 = \begin{array}{c} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array}$$

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$$\frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} = \frac{1}{n} \sum_{v \in V(G)} \begin{array}{c} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \\ \square v \end{array} ; \quad \begin{array}{c} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} = \frac{1}{n} \sum_{v \in V(G)} \begin{array}{c} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \\ \square v \end{array}$$

## EXAMPLE - MANTEL'S THEOREM

### THEOREM (MANTEL 1907)

A triangle-free  $n$ -vertex graph contains at most  $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$  edges.



Assume edges are red and non-edges are blue.

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$$0 \leq \left( 1 - 2 \begin{array}{c} \bullet \\ \text{red edge} \\ \text{yellow square } v \end{array} \right)^2$$





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

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$$0 \leq \left( 1 - 2 \text{  } v \right)^2 = \left( 1 - 4 \text{  } v + 4 \text{  } v + 4 \text{  } v \right)$$

$$\text{  } v \times \text{  } v = \text{  } v + \text{  } v$$









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$$0 \leq \frac{1}{n} \sum_v \left( 1 - 2 \text{  }_v \right)^2 = \frac{1}{n} \sum_v \left( 1 - 4 \text{  }_v + 4 \text{  }_v + 4 \text{  }_v \right)$$

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$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \left( 1 - 2 \text{edge}_v \right)^2 = \frac{1}{n} \sum_v \left( 1 - 4 \text{edge}_v + 4 \text{triangle}_v + 4 \text{triangle}_v \right) \\
 &= 1 - 4 \text{edge} + \frac{4}{3} \text{triangle} + 4 \text{triangle}
 \end{aligned}$$

$$\frac{1}{3} \text{triangle} = \frac{1}{n} \sum_{v \in V(G)} \text{triangle}_v$$



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




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


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 &= 1 - 4 \text{  } + \frac{4}{3} \text{  } + 4 \text{  }
 \end{aligned}$$



$$\text{  } = \frac{2}{3} \text{  } + \frac{1}{3} \text{  } + \text{  }$$







## EXAMPLE - MANTEL'S THEOREM




### THEOREM (MANTEL 1907)

A triangle-free  $n$ -vertex graph contains at most  $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$  edges.

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude   $\leq \frac{1}{2}$ .)

$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \left( 1 - 2 \text{  }_v \right)^2 = \frac{1}{n} \sum_v \left( 1 - 4 \text{  }_v + 4 \text{  }_v + 4 \text{  }_v \right) \\
 &= 1 - 4 \text{  } + \frac{4}{3} \text{  }
 \end{aligned}$$



$$\text{  } = \frac{2}{3} \text{  } + \frac{1}{3} \text{  }$$





## EXAMPLE - MANTEL'S THEOREM



### THEOREM (MANTEL 1907)




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

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude   $\leq \frac{1}{2}$ .)

$$0 \leq \frac{1}{n} \sum_v \left( 1 - 2 \text{  }_v \right)^2 = \frac{1}{n} \sum_v \left( 1 - 4 \text{  }_v + 4 \text{  }_v + 4 \text{  }_v \right)$$

$$= 1 - 4 \text{  } + \frac{4}{3} \text{  }$$

$$0 = 2 \text{  } - \frac{4}{3} \text{  } - \frac{2}{3} \text{  }$$



$$\text{  } = \frac{2}{3} \text{  } + \frac{1}{3} \text{  }$$





## EXAMPLE - MANTEL'S THEOREM



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
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
Assume edges are red and non-edges are blue.




Assume  = 0. (We want to conclude   $\leq \frac{1}{2}$ .)

$$0 \leq \frac{1}{n} \sum_v \left( 1 - 2 \text{  }_v \right)^2 = \frac{1}{n} \sum_v \left( 1 - 4 \text{  }_v + 4 \text{  }_v + 4 \text{  }_v \right)$$

$$= 1 - 4 \text{  } + \frac{4}{3} \text{  }$$

$$= 1 - 2 \text{  } - \frac{2}{3} \text{  }$$

$$0 = 2 \text{  } - \frac{4}{3} \text{  } - \frac{2}{3} \text{  }$$



$$\text{  } = \frac{2}{3} \text{  } + \frac{1}{3} \text{  }$$





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
### THEOREM (MANTEL 1907)



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

$$0 \leq \frac{1}{n} \sum_v \left( 1 - 2 \text{  }_v \right)^2 = \frac{1}{n} \sum_v \left( 1 - 4 \text{  }_v + 4 \text{  }_v + 4 \text{  }_v \right)$$




$$= 1 - 4 \text{  } + \frac{4}{3} \text{  }$$

$$= 1 - 2 \text{  } - \frac{2}{3} \text{  }$$

$$\leq 1 - 2$$



$$0 = 2 \text{  } - \frac{4}{3} \text{  } - \frac{2}{3} \text{  }$$

$$\text{  } = \frac{2}{3} \text{  } + \frac{1}{3} \text{  }$$

## FLAG ALGEBRAS SUMMARY

- Calculations performed over formal linear combinations of graphs
- Evaluated on limits of convergent graph sequences
- Asymptotic results only