Flag Algebras

Ping Hu

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Seminal paper:
Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.
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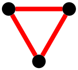
Example (Goodman, Razborov)

If the density of edges is at least $\rho > 0$, what is the minimum density of triangles?

- Designed to attack extremal problems.
- Works well if constraints as well as desired value can be computed by checking small subgraphs (or average over small subgraphs).
- The results are for the limit as graphs get very large.

Flag algebras definitions

Let G be a 2-edge-colored complete graph on n vertices.



The probability that three random vertices in G span a red triangle, i.e. $\#$$/\binom{n}{3}$.

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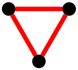

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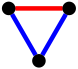

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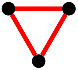

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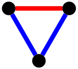

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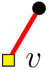

The probability that a random vertex other than v is connected to $v \in V(G)$ by a red edge, i.e., the red degree of v divided by $n-1$.

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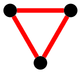

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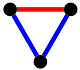

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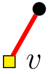

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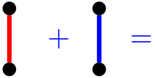

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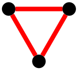

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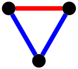

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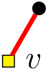
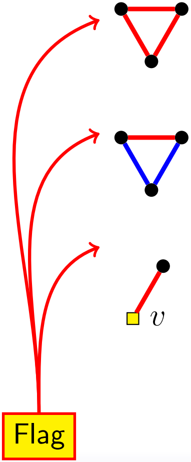


The probability that a random vertex other than v is connected to $v \in V(G)$ by a red edge, i.e., the red degree of v divided by $n-1$.

 $+$  $= 1$

Flag algebras definitions

Let G be a 2-edge-colored complete graph on n vertices.



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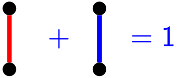
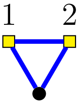
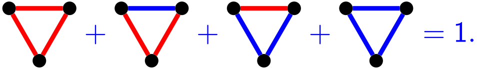




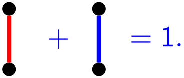
Type is a flag induced by labeled vertices

Flag algebras identities

Let G be a 2-edge-colored complete graph on n vertices. Then

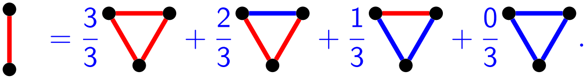


Same as

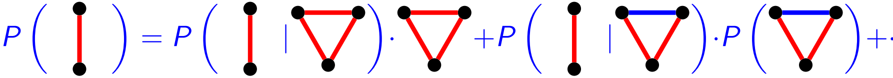


Flag algebras identities

Let G be a 2-edge-colored complete graph on n vertices. Then by the law of total probability



Expanded version:



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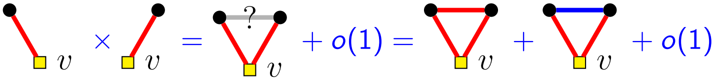

$o(1)$ as $|V(G)| \to \infty$ (will be omitted on next slides)

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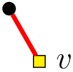

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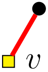

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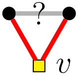
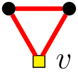


$+\, o(1) =$


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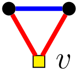

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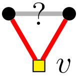

: The probability of choosing two different vertices ...

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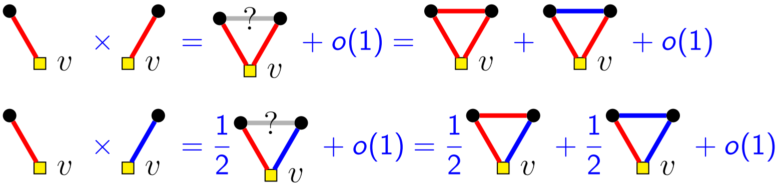

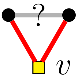

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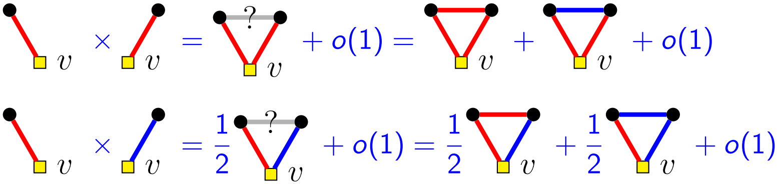

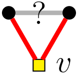

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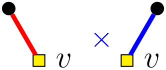

: The probability that choosing two vertices u_1, u_2 other than v gives red vu_1 and blue vu_2.

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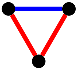
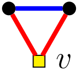
$\frac{1}{3}$  $= \frac{1}{n} \sum_{v \in V(G)}$  v

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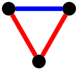
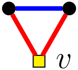
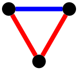
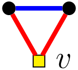
$\frac{1}{3}$  $= \frac{1}{n} \sum_{v \in V(G)}$  v

 $\binom{n}{3} = \sum_{v \in V(G)}$  $v \binom{n-1}{2}$

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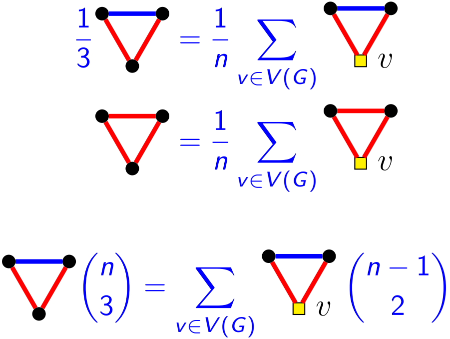

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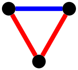
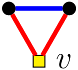
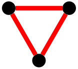
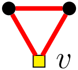
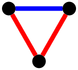
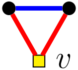
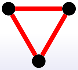
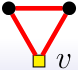
$\frac{1}{3}$  $= \frac{1}{n} \sum_{v \in V(G)}$  v

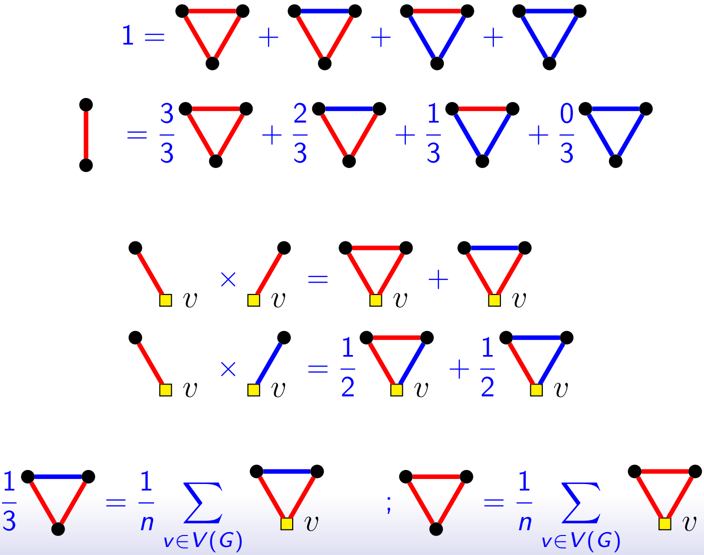
 $= \frac{1}{n} \sum_{v \in V(G)}$  v

 $\binom{n}{3} = \sum_{v \in V(G)}$  $v \binom{n-1}{2}$

 $\binom{n}{3} = \frac{1}{3} \sum_{v \in V(G)}$  $v \binom{n-1}{2}$

Identities Summary

Let G be a 2-edge-colored complete graph on n vertices. Then



Example - Mantel's Theorem

Theorem (Mantel 1907)

A triangle-free n-vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$ edges.

Assume edges are red and non-edges are blue.

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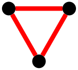
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

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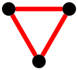
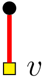
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq$ $\left(1 - 2\right.$  $\left.\right)^2$

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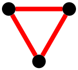
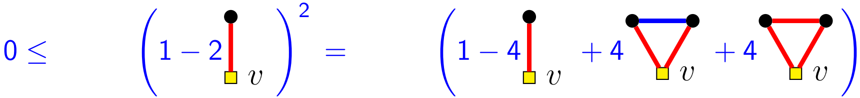
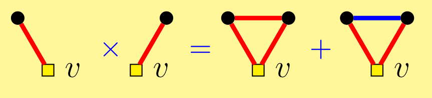
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)





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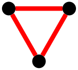
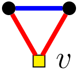
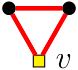
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq \frac{1}{n}\sum_{v}\Big(1 - 2$  $v\Big)^2 = \frac{1}{n}\sum_{v}\Big(1 - 4$  $v + 4$  $v + 4$  $v\Big)$

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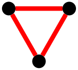
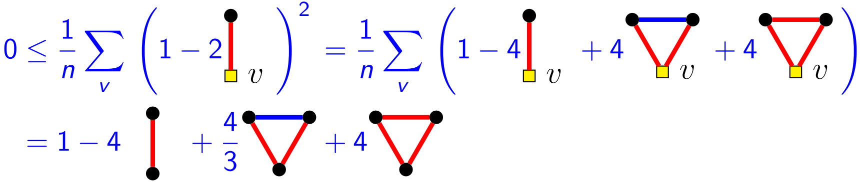
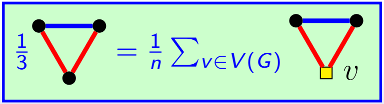
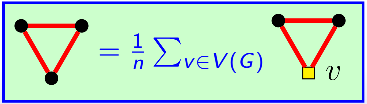
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)







Example - Mantel's Theorem

Theorem (Mantel 1907)

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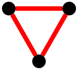
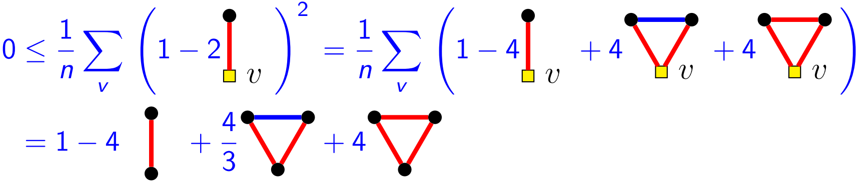
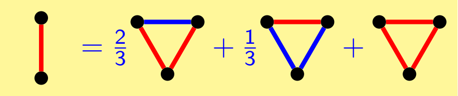
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)





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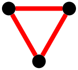
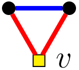
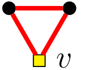
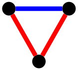
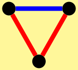
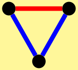
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq \frac{1}{n}\sum_v \Big(1 - 2$  $\Big)^2 = \frac{1}{n}\sum_v \Big(1 - 4$  $+ 4$  $+ 4$  $\Big)$

$= 1 - 4$  $+ \frac{4}{3}$ 

 $= \frac{2}{3}$  $+ \frac{1}{3}$ 

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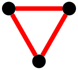
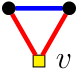
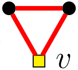
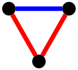
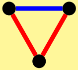
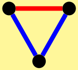
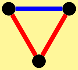
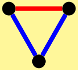
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq \frac{1}{n}\sum_v \Big(1 - 2$  $v \Big)^2 = \frac{1}{n}\sum_v \Big(1 - 4$  $v + 4$  $v + 4$  $v\Big)$

$= 1 - 4$  $+ \frac{4}{3}$ 

$0 = 2$  $- \frac{4}{3}$  $- \frac{2}{3}$ 

 $= \frac{2}{3}$  $+ \frac{1}{3}$ 

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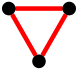
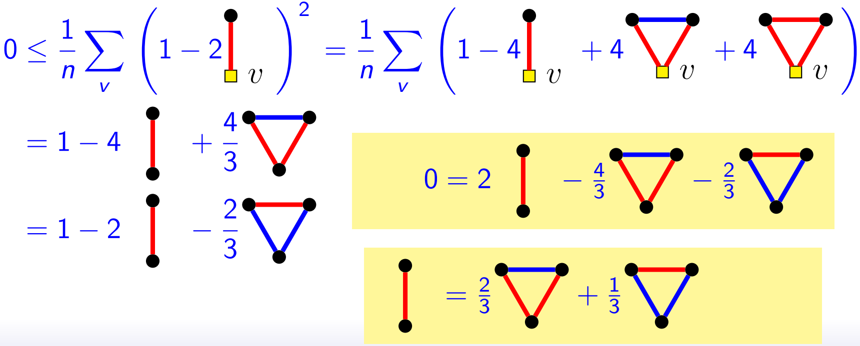
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



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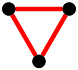
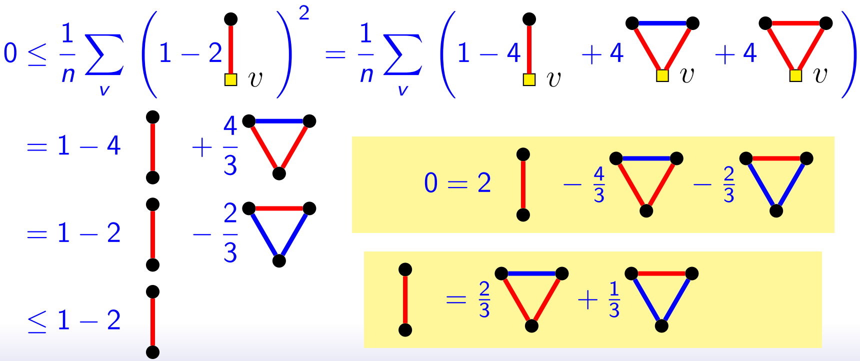
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



Flag algebras summary

- Calculations performed over formal linear combinations of graphs
- Evaluated on limits of convergent graph sequences
- Asymptotic results only