Large Multipartite Subgraphs in H-free Graphs

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Abstract. In this work, we discuss a strengthening of a result of Füredi that every n-vertex K_{r+1}-free graph can be made r-partite by removing at most $T(n,r)-e(G)$ edges, where $T(n,r)=\frac{r-1}{2r}n^2$ denotes the number of edges of the n-vertex r-partite Turán graph. As a corollary, we answer a problem of Sudakov and prove that every K_6-free graph can be made bipartite by removing at most $4n^2/25$ edges. The main tool we use is the flag algebra method applied to locally definied vertex-partitions.

Keywords: Max-Cut · Turán graph · Flag Algebras

1 Introduction

Let $G=(V,E)$ be an n-vertex graph and $r \geq 2$ an integer. Denote by $\mathrm{del_r(G)}$ the minimum size of an edge-subset $X \subseteq E$ such that the graph $G-X$ is r-partite. Note that $\mathrm{del_2(G)}$ is the dual problem to Max-Cut, i.e., finding the largest bipartite subgraph in G. For convenience, we also define $\mathrm{del_1(G)} := \mathrm{e(G)}$.

Our aim is to obtain upper bounds on $\mathrm{del_r(G)}$ and $\mathrm{del_2(G)}$, respectively, when G is a K_{r+1}-free graph, i.e., a graph with no complete subgraph on $r+1$ vertices. A beautiful stability-type argument of Füredi [6] provides the following upper bound on $\mathrm{del_r(G)}$.

Theorem 1. (Füredi [6]). *Fix an integer $r \geq 2$. If G is an n-vertex K_{r+1}-free graph, then* $\mathrm{del_r(G)} \leq \frac{\mathrm{r}-1}{2\mathrm{r}} \cdot \mathrm{n}^2 - \mathrm{e(G)}$.

Note that the number of edges in every K_{r+1}-free graph on n vertices is bounded from above by the number of edges in the Turán graph $T(n,r)$, which is equal

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to $\frac{r-1}{2r} \cdot n^2$. In other words, the result of Füredi can be stated as follows: if a K_{r+1}-free graph is missing t edges to being extremal, then removing at most t edges from it makes it r-partitie.

When the number of edges of G is very close to the extremal value, Theorem 1 was sharpened in [2,7]. Here we focus on a global improvement, and conjecture that Theorem 1 can be strengthened as follows.

Conjecture 1. Fix an integer $r \geq 2$. If G is an n-vertex K_{r+1}-free graph, then $\mathrm{del_r(G)} \leq 0.8 \left(\frac{\mathrm{r}-1}{2\mathrm{r}} \cdot \mathrm{n}^2 - \mathrm{e(G)}\right)$.

If true, Conjecture 1 would be best possible, and we present tight constructions in Sect. 3. Note that for $r \geq 4$, the conjecture does not have a unique extremal example. To provide an evidence for Conjecture 1, we prove it for $r \in \{2, 3, 4\}$.

Theorem 2. *Fix an integer $r \in \{2, 3, 4\}$. If G is an n-vertex K_{r+1}-free graph, then* $\mathrm{del_r(G)} \leq 0.8 \left(\frac{\mathrm{r}-1}{2\mathrm{r}} \cdot \mathrm{n}^2 - \mathrm{e(G)}\right)$.

We also establish the following general improvement on Theorem 1.

Theorem 3. *For every $r \geq 5$ there exists $\varepsilon := \varepsilon(r) > 0$ such that the following holds. If G is an n-vertex K_{r+1}-free graph, then* $\mathrm{del_r(G)} \leq (1 - \varepsilon) \left(\frac{\mathrm{r}-1}{2\mathrm{r}} \cdot \mathrm{n}^2 - \mathrm{e(G)}\right)$.

The bound on $\varepsilon(r)$ we establish monotonically decreases to 0 as r tends to infinity, while Conjecture 1 claims that $\varepsilon(r) = 0.2$ for every r.

A closely related problem inspired by a well-known problem of Erdős on Max-Cuts in dense triangle-free graphs is the following conjecture of Sudakov [9].

Conjecture 2. Fix $r \geq 3$. For every K_{r+1}-free graph G, it holds that

$$\mathrm{del_2(G)} \leq \begin{cases} \frac{(r-1)^2}{4r^2} \cdot n^2 & r \text{ odd, and} \\ \frac{r-2}{4r} \cdot n^2 & r \text{ even.} \end{cases}$$

Note that the conjectured value corresponds to the value of $\mathrm{del_2(T(n,r))}$. Sudakov [9] proved the conjecture for $r = 3$.

Theorem 4. (Sudakov [9]). *An n-vertex K_4-free graph G can be made bipartite by removing $n^2/9$ edges, i.e.,* $\mathrm{del_2(G)} \leq \mathrm{n}^2/9$. *Moreover, if* $\mathrm{del_2(G)} = \mathrm{n}^2/9$, *then G is the Turán graph $T(n, 3)$.*

We prove the conjecture for $r = 5$.

Theorem 5. *If G is an n-vertex K_6-free graph, then* $\mathrm{del_2(G)} \leq 4\mathrm{n}^2/25$. *Moreover, if* $\mathrm{del_2(G)} = 4\mathrm{n}^2/25$, *then G is the Turán graph $T(n, 5)$.*

As we have already mentioned, Erdős [4] made a conjecture on the size of the largest bipartite subgraph in triangle-free graphs. Specifically, he conjectured that $\mathrm{del_2(G)} \leq \mathrm{n}^2/25$ for every triangle-free n-vertex graph G. A result of Erdős, Faudree, Pach, and Spencer [5] states that $\mathrm{del_2(G)} \leq \mathrm{n}^2/18$. Using flag algebras in a manner analogous to the one we use here, an improvement on the last bound was recently announced by Balogh, Clemen, and Lidický [3].

Note that for all the theorems in this section, a straightforward application of the regularity lemma yields the corresponding asymptotic results for H-free graphs, where H is a fixed r-colorable graph.

In our work, we extensively use flag algebras, a versatile tool developed by Razborov [8], applied to K_{r+1}-free graph limits. We use as a convention that unlabeled vertices are depicted as black circles, labeled vertices as yellow squares, and edges as blue lines. Dashed lines indicate that both edge and non-edge are admissible. We write $[\![.]\!]$ to denote the so-called unlabeling/averaging operator.

The rest of this extended abstract is organized as follows: In Sect. 2, we describe an alternative proof of Theorem 1 using flag algebras, which demonstrates the technique we use. In Sect. 3, we examine the set of possible extremal constructions for Conjecture 1, and give a sketch of the proof of Theorem 2 for the case $r = 2$. We conclude the extended abstract by Sect. 4, where we briefly discuss the case $r \geq 3$ as well as the ideas for the proof of Theorem 5.

2 Theorem 1 in Flag Algebras

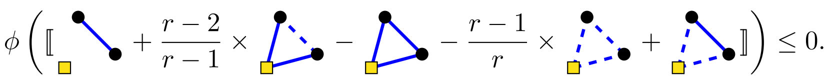
As a warm-up to our flag algebra technique, we present a proof of Theorem 1. Suppose Theorem 1 is false, and let r be the smallest integer for which it fails. Let G be an n-vertex K_{r+1}-free graph G such that $\mathrm{del_r(G)} > \frac{\mathrm{r-1}}{\mathrm{2r}} \cdot \mathrm{n^2 - e(G)}$.

For a vertex $v \in V(G)$, consider an r-partition of $V(G)$ with $A_r := V \setminus N(v)$ being one part, and $(A_1, A_2, \ldots, A_{r-1})$ being an $(r-1)$-partition of $N(v)$ given by Theorem 1 if $r \geq 3$, and $A_1 := N(v)$ in case $r = 2$. Note that if $r = 2$ then $N(v)$ induces no edges in G. It follows that the number of edges inside the parts is at most $e(G[A_r]) + \mathrm{del_{r-1}(G[N(v)])}$, which is as most

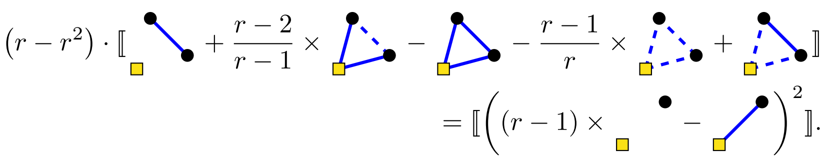
$$e(G[A_r]) + \frac{r-2}{r-1} \cdot \frac{|N(v)|^2}{2} - e(G[N(v)]). \tag{1}$$

On the other hand, this is at least $\mathrm{del_r(G)} > \frac{\mathrm{r-1}}{\mathrm{2r}} \cdot \mathrm{n^2 - e(G)}$. This is in direct contradiction with the following simple flag algebra proposition, which shows that if we choose a vertex v uniformly at random, then the expectation of (1) is at most $\frac{r-1}{2r} \cdot n^2 - e(G)$.

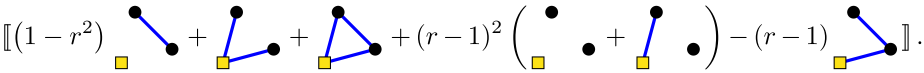
Proposition 1. *Fix $r \geq 2$. If ϕ is a K_{r+1}-free graph limit, then*



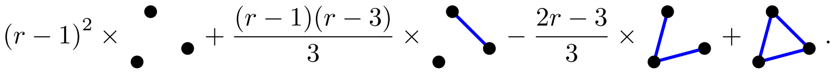
Proof. We will show that the following identity holds for every $r \geq 2$.



Note that the identity immediately proves the statement since the right-hand side is non-negative while $r - r^2 < 0$. Firstly, observe that the left-hand side is equal to

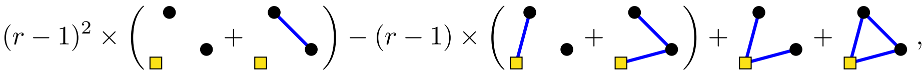


By the definition of $[\![\cdot]\!]$, the previous expression averages to the following:



(2)

On the other hand, the right-hand side of the identity is equal to



which again averages to (2). This finished the proof. □

Proposition 1 and the following lemma yield the statement of Theorem 1.

Lemma 1. *Fix positive integers r, b and ℓ. If G is a K_{r+1}-free graph then its b-blow-up $G[b]$ is K_{r+1}-free and* $\mathrm{del}_\ell(\mathrm{G}[\mathrm{b}]) = \mathrm{b}^2 \cdot \mathrm{del}_\ell(\mathrm{G})$.

An inspection of the just presented proof yields that the bound in Theorem 1 is tight only if G is a Turán graph. Indeed when $G = T(n, r)$, Theorem 1 does not allow to remove any edge. However, this is rather a technical "obstacle" and Conjecture 1 can be seen as a way how to bypass it.

3 Tight Constructions for Conjecture 1

Clearly, Conjecture 1 is tight for Turán graphs since the bound $T(n, r) - e(G)$ does not allow deletion of any edges. When $r = 2$, the complete balanced bipartite graph and a balanced blow-up of C_5 attains the bound $0.8(n^2/4 - e(G))$. Therefore, blow-ups of C_5 behave similarly as a complete bipartite graph with respect to Conjecture 1, and this propagates to larger r.

Given $r \geq 2$, a tight construction for Conjecture 1 can be obtained as follows: Let H be a join of a copies of K_1 and b copies of C_5, where $a + 2b = r$. Let G be a blow-up of H, such that all the vertices corresponding to K_1s have the weight $1/r$ and all the vertices corresponding to C_5s have the weight $2/(5r)$.

When $r \in \{2, 3, 4\}$, we prove the above description of the tight constructions for Theorem 2 is complete, see also Fig. 1.

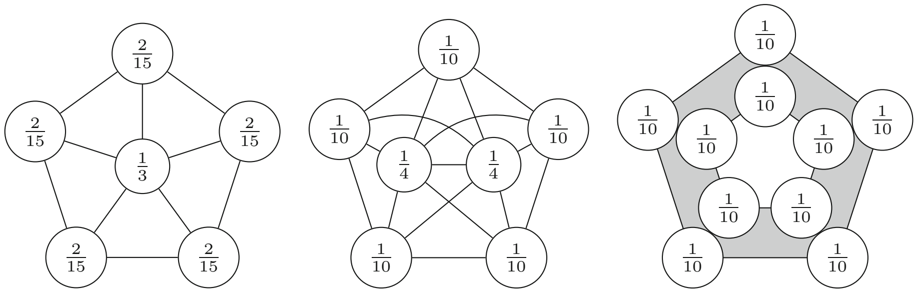


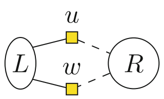
Fig. 1. Non-Turán tight constructions for Theorem 2 when $r = 3$ and $r = 4$.

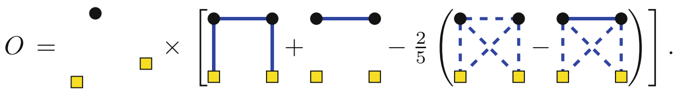
3.1 Proof of Theorem 2 When $r = 2$

Let N be the non-edge type with labels u and w, and let C be the combination of N-flags that expresses the size of the cut (L, R) with $L := N(u) \cup N(v)$ and $R := V \setminus L$. Next, we define

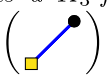
$$O := \overline{K_3^N} \times (C - 0.8(1/2 - d(G))) = \overline{K_3^N} \times \left(C - 0.4(d(\overline{G}) - d(G))\right),$$

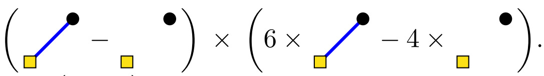
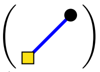
which can be expressed using flag algebras as follows:



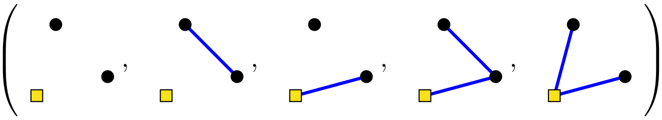


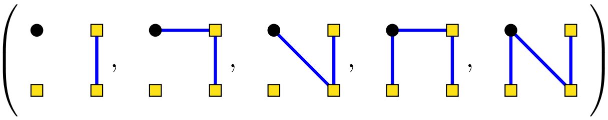
Notice that $\frac{1}{2} - d(G)$ is the density of missing edges to the complete bipartite graph, and $0.8(\frac{1}{2} - d(G))$ is the normalized number of edges we are allowed to delete in Conjecture 1 when $r = 2$. In order to prove Conjecture 1, we need to show that the expression O is non-positive in triangle-free graphs.

Theorem 6. *If ϕ is a K_3-free graph limit, then $\phi([\![O]\!]) \leq 0$. Moreover, if $\phi([\![O]\!]) = 0$, then ϕ^1*  *$\in \{0.4, 0.5\}$ almost surely.*

Proof. First, let $F_1 :=$ . Observe that if $\phi([\![F_1^2]\!]) = 0$ then ϕ^1  $\in \{0.4, 0.5\}$ almost surely.

Next, consider the following two vectors X and Y of σ-flags, where σ is the one-vertex type and the co-cherry type, respectively, and the following 7 linear combinations of flags using X and Y:

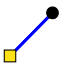
$X =$  , (3)

$Y =$  . (4)

$$F_1 = X \cdot (4, 4, -5, -5, 6), \quad F_4 = Y \cdot (0, 1, -1, 1, -1), \quad F_7 = Y \cdot (6, 1, 1, -4, -4),$$
$$F_2 = X \cdot (6, -9, 0, 0, -6), \quad F_5 = Y \cdot (0, 1, -1, 2, -2), \quad F_8 = Y \cdot (2, -2, -2, 1, 1).$$
$$F_3 = X \cdot (4, 0, -3, -4, 4), \quad F_6 = Y \cdot (0, 2, -2, 1, -1),$$

We express each term as a linear combination of 5-vertex unlabeled flags and establish the following estimate on $[\![O]\!]$ for some non-positive rationals $w_1, w_2, \ldots, w_8$:

$$[\![O]\!] \leq \sum_{i \in \{1,2,\ldots,8\}} w_i \times [\![F_i^2]\!].$$

Hence, $\phi([\![O]\!]) \leq 0$. Moreover, if the equality is attained for some limit ϕ, then $\phi([\![F_i^2]\!]) = 0$ for all $i \in [8]$ by complementary slackness. In particular, we have ϕ^1 () $\in \{0.4, 0.5\}$ for almost every choice of the root. □

Lemma 1 readily translates Theorem 6 to the setting of finite graphs, and a result of Andrásfai, Erdős and Sős [1] yields that the only non-bipartite tight graph in Theorem 2 when $r = 2$ is a balanced blow-up of C_5.

4 Concluding Remarks

An analogous approach to Conjecture 1 when $r = 2$ can be applied to the cases $r = 3$ and $r = 4$, although more locally defined partitions and more sum-of-squares are needed. The proof of Theorem 5 is also very similar, and in fact the simplest form we have found consists only of five sum-of-squares, a natural partition tuned to perform optimally on the corresponding Turán graphs, and an application of Theorem 6.

One of the main reasons why the complexity of the proof grows with r is the increasing number of tight constructions, and it is not obvious how to generalize this approach to all r. Nevertheless, bootstraping from Theorem 6, we establish a much more modest improvement described in Theorem 3.

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